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Performance Engineering for Computational Science

Presentation · October 2018

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Project	Project GESOP View project					

Lattice Boltzmann Methods View project



Regionales Rechenzentrum Erlangen (RRZE), Erlangen, 8. und 9. Okt. 2018



Performance Engineering for Computational Science



Centre Européen de Recherche et de Formation

Avancée en Calcul Scientifique

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U. Ruede



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Outline

- ***** ExaScale computing
 - node efficiency, scalability, and algorithmic complexity
- TerraNeo
 - HHG
 - HyTeG
- **waLBerla**
 - Lattice Boltzmann
 - Rigid Body Dynamics
- **Conclusions**





Part I: Extreme Scale Computing



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1.65

What is it about





Mathematics

and Statistics

CSE

Science

UR, Willcox, K., McInnes, L. C., & Sterck, H. D. (2018). Research and education in **computational science and engineering**. *Siam Review*, *60*(3), 707-754.



What is ExaScale possibly good for?

- ExaScale: 10¹⁸ FLOPS (floating point operations per second)
- 🔐 When we have
 - 1000
 - x 1000
 - x 1000 particles (or pores)
 - each resolved by 1000 cells
- then we still can still execute
 - 1 Mflop per each cell
 - I MFLPOS = 10⁶ FLOPS = the performance of PC in 1990 Pre





Simulation performed with in-house multi-physics framework waLBerla/PE

Preclik, T. & UR (2015). **Ultrascale** simulations of nonsmooth granular dynamics; Computational Particle Mechanics, DOI: 10.1007/s40571-015-0047-6



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10¹⁸ FLOPS ...

At clock rates of 1 Ghz a single stream of operands will produce results at 10⁹/sec

degree of concurrency= 10⁹

- This will be achieved hierarchically
 - ₽ 10⁴ nodes
 - # 10³ cores/node
 - # 10² instructions on the fly/core (vectorization, pipelining, ...)
- We must use them all!
- Energy: 1nJ/Flop
 1 GW for Exascale
- Resilience: MTBF of 100 years per cell phone (= 10 Gflop)
 10⁸ cell phones: MTBF for Exascale = 30 sec





Energy consumption of floating point operations

based on: Ayesha Afzal: *The Cost of Computation: Metrics and Models for Modern Multicore-based Systems in Scientific Computing,* Master Thesis, FAU Erlangen, 2015



8 core Sandy Bridge System - measured through systematic benchmarking see also Georg Hager's talk at PACO 2015

Best values on Green 500 currently convert to 0.1 nJ/Flop: equivalent to 100 MW for ExaFlops performance



Algorithmic energy efficiency

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The first performance question:

What is the minimal cost of solving a PDE?

(such as Poisson's or Stokes' equation in 3D)

asymptotic results of the form

Cost \leq C N^p (Flops?)

with unspecified constant C are inadequate to predict performance

The key goal of numerical analysis:

relate accuracy and cost!

How do we quantify true cost?

(i.e. resources needed)

- Number of flops?
- Memory consumption?
- Memory bandwidth? (aggregated?)
- Communication bandwidth?
- Communication latency?
- Power consumption?



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How do we predict cost?

- What is the cost of solving the discrete Poisson equation?
 - C'mon: it's the mother of all PDE!
 - Yep, there are O(N) algorithms: wavelet, fast multipole, multigrid, ...
 - but what is the best constant published?

for Poisson 2D, second order:

#Flops = **30 N** (Stüben, 1982)

Intel Haswell CPU: 1030.4 GFlops single precision performance

- N=10⁶
- expected time to solution: 3*10-5 sec (microseconds!)
- standard computational practice in 2017 misses this by many orders of magnitude!
- why do we have this enormous gap between theory and practice?
- no, it cannot be explained by "cache effects" alone.
- This talk tries to contribute to the failure analysis.





What is the largest system that we can solve today?

Bergen, B, Hülsemann, F, UR (2005): Is 1.7 · 10¹⁰ unknowns the largest finite element system that can be solved today? Proceedings of SC'05.

- and now, 13 years later?
- we have 400 TByte main memory = 4*10¹⁴ Bytes = 5 vectors each with N=10¹³ double precision elements
- matrix-free implementation necessary
 - even with a sparse matrix format, storing a matrix of dimension N=10¹³ is not possible
- Which algorithm?
 - multigrid
 - Cost = C*N
 - C "moderate", e.g. C=100.
- does it parallelize well on that scale?
- should we worry since $\kappa = O(N^{2/3})$?



Algorithms Matter!

- Solution of Laplace equation in 3D wit N=n³ unknowns
- Direct methods:
 - banded: $\sim n^7 = N^{2.33}$
 - nested dissection: $\sim n^6 = N^2$

- Iterative Methods:
 - Jacobi: ~50 n⁵ = 50 N^{1.66}
 - CG: ~100 n⁴ = 100 N^{1.33}
 - Full Multigrid: ~200 n³= 200 N

Energy per FLOP: 1nJ							
Computer Generation	gigascale: 10 ⁹ terascale: 10 ¹²		petascale: 10 ¹⁵	exascale: 10 ¹⁸			
problem size: DoF=N	10 ⁶	10 ⁹	10 ¹²	10 ¹⁵			
Direct method: 1*N ²	0.278 Wh	278 kWh	278 GWh	278 PWh			
Krylov method: 100*N ^{1.33}	10 Ws	28 Wh 278 kWh		2.77 GWh			
Full Multigrid: 200 N	0.2 Ws	0.056 Wh	56 Wh	56 kWh			
TerraNeo prototype (est. for Juqueen)	0.13 Wh	30 Wh	27 kWh	?			







Part II: Hierarchical Hybrid Grids





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Example: Earth Mantle Convection

Why Mantle Convection?

- driving force for plate tectonics
- mountain building and earthquakes

Why Exascale?

37 mantle has 10¹² km³ inversion and UQ blow up cost

Why TerraNeo?

ultra-scalable and fast sustainable framework

Challenges

computer sciences: software design for future exascale systems *ar* mathematics: HPC performance oriented metrics **geophysics:** model complexity and uncertainty bridging disciplines: integrated co-design



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Meshing of the Mantle with Tets







HHG: A modern mesh-free architecture for FE



Structured refinement of an unstructured base mesh Geometrical Hierarchy: Volume, Face, Edge, Vertex



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HHG Solver for Stokes System Motivated by Earth Mantle convection problem

Gmeiner, Waluga, Stengel, Wohlmuth, UR: Performance and Scalability of Hierarchical Hybrid Multigrid Solvers for Stokes Systems, SISC, 2015.

$$-\nabla \cdot (2\eta \epsilon(\mathbf{u})) + \nabla \boldsymbol{p} = \rho(T)\mathbf{g},$$
$$\nabla \cdot \mathbf{u} = 0,$$
$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla \cdot (\kappa \nabla T) = \gamma.$$

u	velocity
p	dynamic pressure
Т	temperature
ν	viscosity of the material
$\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$	strain rate tensor
ρ	density
$\kappa,\gamma,{f g}$	thermal conductivity,
	heat sources, gravity vector

Scale up to ~10¹³ nodes/ DOFs \Rightarrow resolve the whole Earth Mantle globally with 1km resolution



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Stokes equation: $-\operatorname{div}(\nabla \mathbf{u} - p\mathbf{I}) = \mathbf{f},$ $\operatorname{div} \mathbf{u} = 0$

FEM Discretization:

$$\mathbf{a}(\mathbf{u}_l, \mathbf{v}_l) + \mathbf{b}(\mathbf{v}_l, p_l) = \mathbf{L}(\mathbf{v}_l) \qquad \forall \mathbf{v}_l \in \mathbf{V}_l, \\ \mathbf{b}(\mathbf{u}_l, q_l) - \mathbf{c}(p_l, q_l) = 0 \qquad \forall q_l \in Q_l, \end{cases}$$

with:
$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx, \quad \mathbf{b}(\mathbf{u}, q) := -\int_{\Omega} \operatorname{div} \mathbf{u} \cdot q \, dx$$

Schur-complement formulation:

$$\begin{bmatrix} \mathbf{A}_l & \mathbf{B}_l^\top \\ \mathbf{0} & \mathbf{C}_l + \mathbf{B}_l \mathbf{A}_l^{-1} \mathbf{B}_l^\top \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}}_l \\ \underline{p}_l \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}}_l \\ \mathbf{B}_l \mathbf{A}_l^{-1} \underline{\mathbf{f}}_l \end{bmatrix}$$



Stationary Flow Field







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Exploring the Limits ...

Gmeiner B., Huber M, John L, UR, Wohlmuth, B: A quantitative performance study for Stokes solvers at the extreme scale, Journal of Computational Science, 2016.

- Multigrid with Uzawa smoother
- Optimized for minimal memory consumption 32
 - 10¹³ Unknowns correspond to 80 TByte for the solution vector
 - Juqueen has 450 TByte Memory
 - matrix free implementation essential

- Juqueen nas 450 r Dyte Memory							
matrix free implementation essential							
- maint nee implementation cosenta							
FE System St							
nodes	threads	the larges	TFL	time	time w.c.g.	time c.g. in $\%$	
5	is this	$2.7 \cdot 10^{9}$	10	685.88	678.77	1.04	
40	640	$2.1\cdot 10^{10}$	10	703.69	686.24	2.48	
320	5120	$1.2 \cdot 10^{11}$	10	741.86	709.88	4.31	
2 560	40960	$1.7\cdot 10^{12}$	9	720.24	671.63	6.75	
20 480	327680	$1.1 \cdot 10^{13}$	9	776.09	681.91	12.14	



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Towards systematic performance engineering:

ECM Analysis of multigrid smoother for variable/constant coefficient performance

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Parallel Textbook Efficiency as guiding design principle

Brandt, A. (1998). Barriers to achieving textbook multigrid efficiency (TME) in CFD.

Thomas, J. L., Diskin, B., & Brandt, A. (2003). **Textbook Multigrid Efficiency** for Fluid Simulations*. Annual review of fluid mechanics, 35(1), 317-340.

Gmeiner, UR, Stengel, Waluga, Wohlmuth: Towards **Textbook Efficiency for Parallel Multigrid**, Journal of Numerical Mathematics: Theory, Methods and Applications, 2015

Gmeiner, Huber, John, UR, Wohlmuth, A quantitative **performance study for Stokes solvers** at the extreme scale, Journal of Computational Science, Volume 17, 2016, Pages 509-521



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Textbook Multigrid Efficiency (TME)

"Textbook multigrid efficiency means solving a discrete PDE problem with a computational effort that is only a small (less than 10) multiple of the operation count associated with the discretized equations itself." [Brandt, 98]

- work unit (WU) = single elementary relaxation classical *algorithmic* TME-factor: ops for solution/ ops for work unit
- Here we introduce a new parallel TME-factor:
 - algorithmic efficiency
 - scalability & node efficiency



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ParTME paradigm for parallel performance analysis

- Analyse cost of an elementary relaxation to define cost of work unit (WU) depending on idealized HW
 - micro-kernel benchmarks
 - measure aggregate performance
- Measure parallel solver performance
- Compute ParTME factor as quotient





Parallel TME

 $\mu_{\rm sm}$ # of elementary relaxation steps on single core/sec U # cores

 $U\mu_{\rm sm}$ aggregate peak relaxation performance

 $T_{\rm WU}(N,U) = \frac{N}{U\mu_{\rm sm}}$ idealized time for a work unit

T(N, U) time to solution for N unknowns on U cores

Parallel textbook efficiency factor

$$E_{\text{ParTME}}(N,U) = \frac{T(N,U)}{T_{\text{WU}}(N,U)} = T(N,U)\frac{U\mu_{\text{sm}}}{N}$$

combines algorithmic and implementation efficiency.



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TME Efficiency Analysis: RB-GS Smoother



- This loop should be executed on single SuperMuc core at
 - 720 M updates/sec (in theory peak performance)
 - μ_{sm} = **176 M** updates/sec (*in practice* memory access bottleneck; RB-ordering prohibits vector loads)
- Thus whole SuperMuc should perform
 - $U\mu_{\rm sm}$ = 147456*176M **~ 26 T** (updates/sec)



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Execution-Cache-Memory Model (ECM)

Hager et al. Exploring performance and power properties of modern multi–core chips via simple machine models. Concurrency and Computation: Practice and Experience, 2013.



ECM model for the 15-point stencil on SNB core.

Arrow indicates a 64 Byte cache line transfer.

Run-times represent 8 elementary updates.

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EXERCE REFACS

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ECM: single-chip prediction vs. measurement



Sandy Bridge single-chip performance scaling of the stencil smoothers on 256³ grid points. Measured data and ECM prediction ranges shown.



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ParTME results for Stokes system

Gmeiner, Huber, John, UR, Wohlmuth, A quantitative performance study for Stokes solvers at the extreme scale, Journal of Computational Science, vol.17, 2016, pp. 509-521

Drzisga, D., John, L., UR, Wohlmuth, B., & Zulehner, W. (2018). On the Analysis of Block Smoothers for Saddle Point Problems. *SIAM Journal on Matrix Analysis and Applications*, *39*(2), 932-960.

	TME	ParTME				
DoFs		8.2×10 ⁶	5.3×10 ⁸	4.3×10 ⁹	3.4×10 ¹⁰	2.7×10 ¹¹
cores		1	192	1536	12288	98304
FMG-1V(2,2)	9.07	43.43	150	200	278	500

- variable V-cycle, Uzawa-type smoother
- TME factor <10
- ParTME factor >200
- coarse grid solver becomes a bottleneck

Nevertheless we are reaching **10¹³** unknowns.





Redesigning the HHG prototype Hybrid Tetrahedral Grids - HyTeG

- Geometric and algebraic
 multigrid solvers with scalable
 complexity
 - Scalable solvers
 - Robust higher order discretization (Fast4HHO, EoCoE)
 - Scalable & Robust Coarse Grid solvers, MUMPS, …
 - ...
- Advanced algorithms with potential for exascale computing
 - Matrix-free Methods (EoCoE)
 - Algorithm Based Fault Tolerance, ABFT

- Software for ExaScale
 Computing
 - TerraNeo Project
 - HyTeG Software (HHG)

Bauer, S., Mohr, M., UR, Weismüller, J., Wittmann, M., & Wohlmuth, B. (2017). A two-scale approach for efficient **on-the-fly operator assembly** in massively parallel high performance multigrid codes. *Applied Numerical Mathematics*, *122*, 14-38.

Bauer, S., Drzisga, D., Mohr, M., UR, Waluga, C., & Wohlmuth, B. (2017). A stencil scaling approach for accelerating **matrix-free** finite element implementations. *arXiv preprint arXiv:1709.06793 (submitted to SISC)*

Kohl, N., Thönnes, D., Drzisga, D., Bartuschat, D., UR (2018). The **HyTeG** finite-element software framework for scalable multigrid solvers. *International Journal of Parallel, Emergent and Distributed Systems*, 1-20.

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HyTeG: Hybrid tetrahedral grids

- Generalizes HHG mesh structures for arbitrary discretizations
- Structured-withinunstructured mesh





Example element types

(e) DG1 (C) element



(d) FV/DG0 element

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(f) DG1 (E) element

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Part III: Lattice Boltzmann

Succi, S. (2001). The lattice Boltzmann equation: for fluid dynamics and beyond. Oxford university press. Feichtinger, C., Donath, S., Köstler, H., Götz, J., & Rüde, U. (2011). WaLBerla: HPC software design for computational engineering simulations. Journal of Computational Science, 2(2), 105-112.



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Single Node Performance

JUQUEEN

SuperMUC



Pohl, T., Deserno, F., Thürey, N., UR, Lammers, P., Wellein, G., & Zeiser, T. (2004). Performance evaluation of parallel largescale lattice Boltzmann applications on three supercomputing architectures. Proceedings of the 2004 ACM/IEEE conference on Supercomputing (p. 21). IEEE Computer Society.

Donath, S., Iglberger, K., Wellein, G., Zeiser, T., Nitsure, A., & UR (2008). Performance comparison of different parallel lattice Boltzmann implementations on multi-core multi-socket systems. International Journal of Computational Science and *Engineering*, *4*(1), 3-11.



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Weak scaling for TRT lid driven cavity - uniform grids



Domain Partitioning and Parallelization





Performance on Coronary Arteries Geometry



Color coded proc assignment

Godenschwager, C., Schornbaum, F., Bauer, M., Köstler, H., & UR (2013). A framework for hybrid parallel flow simulations with a trillion cells in complex geometries. In *Proceedings* of SC13: International Conference for High Performance Computing, Networking, Storage and Analysis (p. 35). ACM.





Weak scaling 458,752 cores of JUQUEEN over a trillion (10¹²) fluid lattice cells

Strong scaling

32,768 cores of SuperMUC

- cell sizes of 0.1 mm
- 2.1 million fluid cells
- 6000 time steps per second



Adaptive Mesh Refinement and Load Balancing

Isaac, T., Burstedde, C., Wilcox, L. C., & Ghattas, O. (2015). Recursive algorithms for distributed forests of octrees. SIAM Journal on Scientific Computing, 37(5), C497-C531. Meyerhenke, H., Monien, B., & Sauerwald, T. (2009). A new diffusion-based multilevel algorithm for computing graph partitions. Journal of Parallel and Distributed Computing, 69(9), 750-761.

Schornbaum, F., UR (2016). Massively Parallel Algorithms for the Lattice Boltzmann Method on **Nonuniform Grids**. SIAM Journal on Scientific Computing, 38(2), C96-C126.

Schornbaum, F., UR (2018). Extreme-scale block-structured **adaptive mesh** refinement. *SIAM Journal on Scientific Computing*, *40*(3), C358-C387.

Ultrascalable Algorithms - Ulrich Rüde



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IROPÉEN DE RECHERCHE ET DE FORMATION AVANCÉE EN CALCUL SCIENTIFIQUE



AMR Performance

• JUQUEEN – space filling curve: Morton







Ultrascalable Algorithms -

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AMR Performance









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Application Example:

Direct simulation of complex fluids



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Simulation of suspended particle transport





shear stress at bed surface



Preclik, T., Schruff, T., Frings, R., & Rüde, U. (2017, August). Fully Resolved Simulations of **Dune Formation in Riverbeds**. In *High Performance Computing: 32nd International Conference, ISC High Performance 2017, Frankfurt, Germany, June 18-22, 2017, Proceedings* (Vol. 10266, p. 3). Springer.



Pore Scale Computation — Ulrich Rüde



Simulation of suspended particle transport



Sedimentation and fluidized beds



ROPÉEN DE RECHERCHE ET DE FORMATION AVANCÉE EN I



- 3 levels mesh refinement 22
- 3800 spherical particles 32
- Galileo number 50 32
- 128 processes
- 1024-4000 blocks 32

40

Block size 32³



Pore Scale Computation — Ulrich Rüde

Study of particle shape effects



prolate

spherical

oblate

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Rettinger, C., UR (2017). A comparative study of **fluid-particle coupling** methods for fully resolved lattice Boltzmann simulations. *Computers & Fluids, 154, 74-89.*

Rettinger, C., UR (2018). A coupled lattice Boltzmann method and discrete element method for discrete particle simulations of **particulate flows**. *Computers & Fluids*.

Preclik, T., UR (2015). Ultrascale simulations of non-smooth **granular dynamics**. *Computational Particle Mechanics*, *2*(2), 173-196.

Eibl, S., UR (2018). A Systematic Comparison of Dynamic **Load Balancing** Algorithms for Massively Parallel Rigid Particle Dynamics. *arXiv preprint arXiv:1808.00829*.





Why are these codes so fast?



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Pessiptimizing the Performance

with greetings from D. Bailey's: Twelve Ways to Fool the Masses...

- traverse memory in unfavorable order, ignore caches, use many small MPI messages
 - 10¹³ → 10¹² unknowns
- Do not use a matrix-free implementation: a single multiplication with the mass and stiffness matrix can easily cost 50 memory accesses
 - 10¹² → 10¹¹ unknowns
- Write "generic code": implement unstructured grids, pointers, using extensively indirect memory access
 10¹¹ -> 10¹⁰ unknowns
- algorithmic overhead: check convergence redundantly, use an expensive error estimator, etc.
 - 10¹⁰ → 10⁹ unknowns
 (... still a large system ...)



Why is optimization hard?

with four strong jet engines

Would you want to propel a Superjumbo

Our usual Software Design

Moderately Parallel Computing

or with 1,000,000 blow dryer fans?







The End of Moore's Law

We are used to 1000x improvement per decade through:

Transistor Scaling - this will come to a halt

We must change strategy:

- Architectural "improvements" (GPU, manycore)
- **#** Algorithms
- Implementation

Moore's *deflation of computational cost* has dictated the economics of computing - and now this will gradually change.





In the coming decade ...

- absolute "performance" instead of "scalability"
- We will gradually refactor/redesign many of today's application codes
 - it is not a question of cost but of development time
- Gradual shift of investments from HW to SW
 - slowdown of HW investment cycles
 - performance improvements through "performance engineering"





Computational Science is done in Teams

- **Pr.-Ing.** Dominik Bartuschat
- Martin Bauer, M.Sc. (hons)
- 37 Dr. Regina Degenhardt
- Sebastian Eibl, M. Sc.
- Pipl. Inf. Christian Godenschwager
- Marco Heisig, M.Sc.(hons)
- PD Dr.-Ing. Harald Köstler
- 🔐 Nils Kohl, M. Sc.
- Sebastian Kuckuk, M. Sc.
- Christoph Rettinger, M.Sc.(hons)
- 🔐 Jonas Schmitt, M. Sc.
- Pipl.-Inf. Florian Schornbaum
- Dominik Schuster, M. Sc.
- 🔐 Dominik Thönnes, M. Sc.

- 🔐 Dr.-Ing. Benjamin Bergen
- Dr.-Ing. Simon Bogner
- Pr.-Ing. Stefan Donath
- Dr.-Ing. Jan Eitzinger
- Pr.-Ing. Uwe Fabricius
- 🔐 Dr. rer. nat. Ehsan Fattahi
- Pr.-Ing. Christian Feichtinger
- 🔐 Dr.-Ing. Björn Gmeiner
- 🔐 Dr.-Ing. Jan Götz
- 🔐 Dr.-Ing. Tobias Gradl
- 🔐 Dr.-Ing. Klaus Iglberger
- **Pr.-Ing.** Markus Kowarschik
- Dr.-Ing. Christian Kuschel
- Pr.-Ing. Marcus Mohr
- 🔐 Dr.-Ing. Kristina Pickl
- 🔐 Dr.-Ing. Tobias Preclik
- 🔐 Dr.-Ing. Thomas Pohl
- Pr.-Ing. Daniel Ritter
- R Dr.-Ing. Markus Stürmer
- 🔐 Dr.-Ing. Nils Thürey
- Ulrich Rüde



Coupled Flow for ExaScale





Thank you for your attention!



Bogner, S., & UR. (2013). Simulation of floating bodies with the lattice Boltzmann method. *Computers & Mathematics with Applications*, *65*(6), 901-913.

Anderl, D., Bogner, S., Rauh, C., UR, & Delgado, A. (2014). Free surface lattice Boltzmann with enhanced bubble model. *Computers & Mathematics with Applications*, 67(2), 331-339.

Bogner, S. Harting, J., & UR (2017). Simulation of liquid-gas-solid flow with a free surface lattice Boltzmann method. Submitted.



Simulation und Vorhersagbarkeit — Ulr

Ulrich Rüde

